

III. Renaissance Europe

In tracing the development of mathematics through the ages, it is tempting to move immediately from the Greek civilization to Renaissance Europe. Roughly 1000 years however passed between these times – the so called Middle Ages (or Dark Ages) – and it would not suffice to neglect the achievements of the people of this time. Before we cover their mathematical activity, we should first give some background.

Section 1: The Middle Ages

Ages in the history of man do not start and stop as sharply as we imply when we summarize them. It is impossible to say exactly when the Middle Ages ended and the Renaissance began. However, for the purposes of broadly describing large periods of time, we shall allow ourselves to characterize these times in just this way. Prior to the Middle Ages, the Western world was in some ways one enormous civilization, comprising the Romans, Egyptians, Persians, and Greeks. Trade between nations was prevalent, and knowledge, goods, and services were easily disseminated throughout the “known world”. However, with the fall of Rome in 476, the intellectual achievements of the West slowed to a trickle. The Chinese, Indians/Hindus, and Persians/Arabians took up the mantle of preserving and fostering academic pursuits. Of primary interest at the beginning this chapter are the Persians.

After the rise of Islam following Mohammed’s flight to Medina in 622, the myriad Arabian and Persian tribes united under a single nation and purpose. This new culture quickly spread throughout the Middle East as the Moslems set out to conquer and convert the known world. At its apex, the Persian Empire reached as far east as the borders of India and as far west as Spain. Meanwhile in Europe, the political, economic, and cultural power began a slow migration to the north. Empires ceased to exist in the way they had previously. The Roman Empire had broken into eastern and western halves in 305, and although the Eastern Roman Empire existed in some form or another throughout the next 1100 years, its impact was greatly reduced. After the fall of the Western Roman Empire (centered at Rome), the political, economic, and cultural power began migrating north. The First Holy Roman Empire, under Charlemagne, existed from 800-814 in what is now France, and then the Second Holy Roman Empire (in what is now Germany) dominated the political landscape from 960 until broken up by Napoleon in 1806. But it bears repeating that these empires were mere shadows of the Roman, Egyptian, and Babylonian Empires of old, and the rising Persian Empire surrounding the Mediterranean Sea.

In 1085, the Christians conquered Toledo (in present day Spain) from the Moors (Persians). In the 200 years that followed, Christians undertook the Crusades – eight separate attacks on the holy lands of the Middle East in an attempt to reclaim this land from the Moslems. This was followed by an influx of Christian scholars to Moslem cities to acquire learning. As we have noted, it was the Moslems that sustained ancient knowledge in general and mathematics in particular. After the capture of a city, the Persians set about translating all academic works into Arabic. In this way, many works that would have been forever lost were preserved.

In addition to capturing a city and translating their knowledge in Arabic, several Persian mathematicians were quite accomplished in their own right. Tabit ibn Qorra (826-901) produced the best Arabic translation of Euclid's *Elements* and wrote on many diverse topics such as astronomy, conic sections, magic squares, and amicable numbers. Abu l-Wefa (940-998) is well known for his translation of Diophantus' *Arithmetica*, but he also did extensive work in trigonometry and is credited with the introduction of the tangent function. A couple of other Persians, Abu Kamil (850-930) and al-Karkhi (953-1029), were accomplished in algebra. But perhaps the two most original and important Arabic mathematicians were Mohammed ibn Musa al-Khowarizmi and Omar Khayyam.

Omar Khayyam (1050-1130) produced a geometrical solution to cubic equations, and while he rejected negative roots and sometimes didn't find all positive solutions, it was still a remarkable achievement in its generality. Al-Khowarizmi (780-850), perhaps the most influential Persian mathematician, wrote a treatise on algebra and a book on the Hindu numerals (which were much more utilitarian than the Roman numerals), and as we shall soon see, both texts had quite an impact in Europe centuries later. In the early days of the Moslem culture, Arabians wrote out all their numbers in words. As they overtook cities and countries, sometimes local numeral systems were used. Eventually however, led by al-Khowarizmi, Moslems adopted the Hindu numerals. Al-Khowarizmi's most influential work was *Hisab al-jabr w'al-muqabalah* (which roughly translates as "The Science of Reunion and the Opposition"). Interestingly, it is from this title – and the word *al-jabr* in particular – that we get the word *algebra* for the science of equations. Also, since al-Khowarizmi's book contained many detailed explanations of how to solve equations, his name became synonymous with how to solve equations. When it was eventually translated into Latin, *al-jabr* began with the phrase "Spoken has Algoritmi..." (hence his name had become Algoritmi), and eventually this gave us the word *algorithm* for "the art of calculating in any particular way."

[This reminds me of one further bit of etymology that is fascinating. The origin of the word *sine* begins with the Indian Aryabhata (476-550). He used the words *ardha-jya* ("half chord") and *jya-ardha* ("chord half") and then abbreviated this to the term *jya*. It was from this, that the Arabs phonetically derived the word *jiba*, but following their practice of omitting vowels, this was written as *jb*. Unfortunately, *jiba* is (aside from its technical meaning) nonsense in Arabic. So when later Moslem writers came across *jb*, they substituted *jaib* (instead of the correct *jiba*), which happens to be a perfectly good Arabic word meaning "cove" or "bay." Once Europeans began their translations, *jaib* was replaced with its Latin equivalent *sinus*, from we got *sine*.]

While all of this was going on in the Persian Empire, there were some advances being made in Europe as well. Some of the greatest achievements of the Middle Ages were translating earlier works into Latin (such as Qorra's translation of *Elements*, l-Wefa's translation of *Arithmetica*, and al-Khwarizmi's *al-jabr*) and the founding of the modern university (in cities such as Paris, Oxford, Cambridge, Padua, and Naples). The founding of these institutions laid the foundation for the grand explosion of mathematical activity that was to come during the Renaissance (and has continued until today). Neither of these achievements was specifically mathematical however. There were many factors that contributed to the lack of mathematical progress in Europe during the Middle Ages, but one of the most important was the inadequate

number system in use. As mathematics became more and more sophisticated, arithmetic using the Roman numerals became more cumbersome, but the adoption of Hindu-Arabic numerals (that we use today) was a slow process. Eventually, what they needed was a strong advocate and they found one in Fibonacci of Pisa.

Section 2: Fibonacci

The preeminent mathematician of the Middle Ages was Leonardo Fibonacci of Pisa. Fibonacci was born in 1175 in Pisa where his father was involved with the mercantile business. [ADD] It was in his famous work *Liber abaci* that Hindu-Arabic numerals received their strongest support to date. In fact, the text opens with the following passage:

These are the nice figures of the Indians

9 8 7 6 5 4 3 2 1

With these nine figures, and with the sign 0...any number can be written as will below be demonstrated.

In the fifteen chapters that follow, Fibonacci explains the reading and writing of these new numerals, methods of calculation, computation of square and cube roots, and the solution of linear and quadratic equations. Despite the advantages of the Hindu-Arabic numerals over their more common Roman competitors, their acceptance was not immediate. It took until the 15th century, with the advent of printing, for their forms to become standardized and their use to become commonplace.

Probably Fibonacci's most famous contribution to mathematics as a whole, aside from his advocacy of Hindu-Arabic numerals, is the sequence that bears his name. The sequence of numbers 1, 1, 2, 3, 5, 8, ... is perhaps the most important and most studied sequence of all time. It is formed by beginning with two consecutive ones (or in more generality any two integers), and then each successive term is the sum of the two previous terms. The sequence is produced from the following problem in *Liber abaci*:

How many pairs of rabbits can be produced from a single pair in a year if every month each pair begets a new pair, which from the second month on, becomes reproductive?

Of course, if this origin was the only "application" of the Fibonacci sequence, it would not have endured or enjoyed such popularity. To the contrary, there are numerous demonstrations and application of the sequence in nature, science and the arts. In mathematics, the sequence appears in Pascal's triangle, the binomial formula, probability, the golden ratio, magic squares, and many other topics.

Eventually, Fibonacci's talents came to the attention of the Emperor Fredrick II. He invited Fibonacci to court to participate in a mathematical tournament. In this venue, Fibonacci's star truly began to rise. He was presented with three problems by one of the

emperor's advisors and his answers demonstrated his prowess. The second problem was to find a solution to the cubic equation $x^3 + 2x^2 + 10x = 20$. Fibonacci attempted a proof that no root could be of the form $\sqrt{a + \sqrt{b}}$ and then approximated a root correctly to nine places. He published this result without any accompanying discussion in a work entitled *Flos* ("flower") and there is great curiosity how he arrived at this solution.

Section 3: Italy and the Theory of Equations

The middle of the 15th century saw the start of the European Renaissance in art and learning. Many factors contributed to this rebirth, including the fall of Constantinople to the Turks in 1453. Many scholars fled the remnants of the Byzantine Empire and sailed west through the Mediterranean Sea landing in Italy and Spain. They brought with them knowledge and treasure from the ancient Greek civilization, and sparked a renewed curiosity on the continent. Other factors that enabled such a dramatic shift in academic priorities were the invention of the printing press and rapid increase in explorations.

By the early part of the 16th century, academic endeavors including mathematics were given increased attention. A prime example of this is the fascination with cubic and quartic equations by Italian number theorists. In 1515 Scipione del Ferro (1465-1526) algebraically solved cubic equations of the form $x^3 + mx = n$. He did not publish this result, but revealed the secret to one of his students. When, in 1535, Nicola Fontana (also known as "Tartaglia") claimed to have discovered an algebraic solution to cubic equations of the form $x^3 + px^2 = n$, del Ferro's student challenged Tartaglia to a mathematical duel. In preparation for the public contest, Tartaglia diligently applied himself and a few days before the event found an algebraic solution to cubic equations lacking a quadratic term. He was victorious in the contest and became quite famous.

In 1545, Girolamo Cardano published the great Latin treatise *Ars magna*. In it, he revealed Tartaglia's secret to solving cubics, having wrested it from Tartaglia under a solemn pledge of secrecy. A student of Cardano denied this claim, insisting that this information was received from del Ferro. The dispute became quite virulent and Tartaglia was lucky to have survived it. Ludovico Ferrari was Cardano's student and his most ardent defender. He was also an accomplished mathematician in his own right. In 1540, he discovered a general algebraic solution for the quartic (or biquadratic) equation. This, too, was published in *Ars magna*.

An interesting byproduct of this intense push for algebraic solutions to polynomial equations was the growing exposure of Europe to imaginary numbers. Since ancient times there were always "disallowed" numbers. We have seen zero, negative numbers, and irrational numbers all progress through this stigmatized state to eventual acceptance. However, square roots of negative numbers were still taboo and "nonsense". In fact, in *Ars magna*, Cardano resisted even negative coefficients in his equations. For example, he willingly handles quadratic equations of the type "square plus cosa equals number" like $x^2 + 3x = 40$, yet he refrains from dealing with equations of the form "square minus cosa equals number" such as $x^2 - 3x = 40$. Instead, he describes a procedure for solving the equivalent $x^2 = 3x + 40$. Nevertheless, it was

Cardano who took the first peek into a world where imaginary numbers existed. In *Ars magna*, he posed the following problem and solution:

Someone says to you, divide 10 into two parts, one of which multiplied into the other shall produce 40. This is impossible to solve. Nevertheless, we shall solve it in this fashion...

He then suggested $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ as the two required numbers. Indeed,

$$(5 + \sqrt{-15}) + (5 - \sqrt{-15}) = 10, \text{ and}$$

$$(5 + \sqrt{-15}) \cdot (5 - \sqrt{-15}) = 25 - 5\sqrt{-15} + 5\sqrt{-15} - (-15) = 40.$$

Admittedly, Cardano called this solution “puzzling” and said it was “as subtle as it is useless.” Fortunately, this development was far from useless.

A seemingly paradoxical situation developed soon after Cardano’s controversial publication was released. As cubic after cubic was now being solved, something strange occurred. Consider the cubic $x^3 - 78x = 220$ for example. This is of the form “cube minus cosa equals number” was not one that was addressed, even though it was similar to del Ferro’s $x^3 + mx = n$ if we let $m = -78$ and $n = 220$. However, if Cardano’s cubic formula is applied to this equation, we come across the number $\sqrt{-5476}$. This stopped all calculations, and rendered this equation “unsolvable”. The real headache for mathematicians here was that the cubic $x^3 - 78x = 220$ actually has three *real* solutions, namely 10, $-5 + \sqrt{3}$, and $-5 - \sqrt{3}$!

This remained unresolved until Rafael Bombelli in 1572. In his text *Algebra*, he suggested that square roots of negative numbers could be introduced, at least temporarily, in solving cubic equations. In so doing, Bombelli was able to utilize $\sqrt{-1}$ and find the solution $x = 10$ to the equation $x^3 - 78x = 220$. This approach, called a “wild thought” by Bombelli himself, amazed mathematicians of the day. However, they were not convinced in the reality that such numbers existed for quite some time. In fact, Bombelli even said that this approach seemed to work by magic. In 1637, when René Descartes published *La Geometrie*, he referred to numbers such as $\sqrt{-9}$ as “imaginary”. They would remain on the fringe of acceptance, yet outside of it, until the 18th century when Leonhard Euler introduced the symbol i for $\sqrt{-1}$.

Section 4: Others

Another important change during the Renaissance was the improved and unified algebraic symbolism. In this area, none stood larger than the French mathematician François Viète. Born in 1540, Viète was primarily a lawyer and a member of parliament. As his wealth increased, however, he spent more and more of his leisure time studying mathematics. In particular, he was interested in cryptography, trigonometry, and algebra.

During a war with Spain, Viète decoded many intercepted Spanish letters (containing a complex code of over 400 characters) for King Henry IV of France. King Phillip II of Spain was so certain that the code was undecipherable, that when it was broken, he complained to the Pope that the French were using black magic against his country.

In 1593, after an ambassador from the Low Countries (The Netherlands) claimed that France had no mathematician able to solve a particular problem (posed by a countryman Romanus), Viète was summoned to the court of Henry IV and shown the 45th degree equation. Within a few minutes, Viète produced two roots, and later added 21 more. Typical of the time, the negative roots escaped him. Upon hearing this feat, Romanus traveled to meet Viète and a warm, lifelong friendship developed.

His political life was routinely besieged by turmoil. He was allied with the Protestant Reformation in France, and therefore he had many Catholic (and powerful) enemies. He was also highly critical of the Gregorian calendar prepared by Cristof Clavis, and unleashed a long series of bitter attacks against him. This did not repair his image in the eyes of the church.

Mathematically, he is most known for his algebraic symbolism. He culled from the most respected texts of the previous 1500 years, and made many improvements. He used vowels of the alphabet for unknown quantities and consonants for known ones. This system received only a mild revision by Descartes in 1637 to produce our current system (letters at the end of the alphabet for unknowns, at the beginning for knowns). Instead of a different symbol for each power of a quantity or unknown, Viète made the change to a single variable properly qualified. By this time other standards were becoming common in algebra. In 1489 Widman introduced the + and - signs to represent positive and negative quantities (respectively), and in 1557 Robert Recorde used the two equal length parallel lines “=” to represent equality. He explained this symbol saying: “because noe 2 thynges can be moare equalle.” These changes in symbolism along with the new Hindu-Arabic numerals and the printing press made it possible to communicate new mathematics very effectively, which enabled rapid growth in the field.

Finally, a few words should be written about Galileo and Kepler. As mentioned several times, the Renaissance was surely an amazing time. Intellectual curiosity and scientific inquiry, long dormant, had awoken. Many old ideas were being revised or discarded altogether. Two classic examples of this are Galileo Galilei (1564-1642) and Johann Kepler (1571-1630). Both men were accomplished in many scientific fields, including mathematics, but their primary claims to fame lie in astronomy. Galileo taught mathematics at the University of Pisa and later – after he was forced to resign after disproving Aristotle’s claim that heavier bodies fall faster than lighter ones – at the University of Padua. It was while at Padua, that Galileo made a startling discovery with his new invention the telescope. On January 7, 1610, he observed two “stars” to the east of Jupiter and one to the west. The next night, all three were on the west. A few nights later he found a fourth object and realized that these objects were orbiting around Jupiter. This confirmed Copernicus’ theory that objects could revolve around larger ones...in particular, the Earth could revolve around the sun. Of course, the Church at the time believed that the Earth was the center of the universe and therefore could not move. Eventually, Galileo was summoned before the Inquisition and, under threat of torture, forced to recant his findings. Legend has it that as he stood up to leave, he quietly said, “The Earth *does* move all the same.” It was not the

brightest moment for the Church. Galileo was a devout Catholic his entire life. He believed that the Bible was not intended to teach us scientific truths that we could learn for ourselves, but rather spiritual truths we could not.

Kepler is a model of persistence. He fervently believed Copernicus' theory that the planets revolved around the sun and was an assistant to the famous astronomer Tycho Brahe (1546-1601). Brahe had collected volumes of data detailing the motions of the planets and Kepler was determined to discover the nature of their motion. He had to make an initial guess at their orbit and then repeatedly refine his guess to conform to Brahe's data. After laboring on this project for twenty-one years, he was able to finally state his three Laws of Planetary Motion. This remains one of the great inductive discoveries ever made in science.